Filomat 29:5 (2015), 1149–1162
DOI 10.2298/FIL1505149S

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Estimation of Stress-Strength Reliability Using Record Ranked Set Sampling Scheme from the Exponential Distribution

Mahdi Salehi, Jafar Ahmadi*

Department of Statistics, Ordered and Spatial Data Center of Excellence,
Ferdowsi University of Mashhad, P.O. Box 1159, Mashhad, 91775 Iran

Abstract. In this paper, point and interval estimation of stress-strength reliability based on upper *record ranked set sampling* (*RRSS*) from one-parameter exponential distribution are considered. Maximum likelihood estimator (*MLE*) as well as the uniformly minimum variance unbiased estimator (*UMVUE*) of stress-strength parameter are derived and their performance are studied. Also, some confidence intervals for stress-strength parameter based on upper *RRSS* are constructed and then compared on the basis of a simulation study. Finally, a data set has been analyzed for illustrative purposes.

1. Introduction

Let the random variable X represent the *stress* experienced by the component and the random variable Y stand for the *strength* of the component available to overcome the stress. If the stress exceeds the strength, i.e. $X > Y$, the component would fail. Thus, reliability is defined as the probability of not failing or $\Pr(X < Y)$. In reliability context, the parameter $R := \Pr(X < Y)$ is called *stress-strength* reliability. Parametric and non-parametric inferences on R for several specific distributions of X and Y under different sampling schemes have been found in the literature. It seems that Birnbaum and McCarty (1958) was the first paper with $\Pr(X < Y)$ in its title. They obtained a non-parametric upper confidence bound for $\Pr(Y < X)$. Owen et al. (1964) studied the stress-strength R under parametric assumptions on X and Y. They constructed confidence limits for R when X and Y are dependent or independent normally distributed random variables. There are several works on the inference procedures for R based on complete and incomplete data from X and Y samples. We refer the readers to Kotz et al. (2003) and references therein for some applications of R. This book collects and digests theoretical and practical results on the theory and applications of the stress-strength relationships in industrial and economic systems up to 2003. Kundu and Raqab (2009) considered the estimation of the stress-strength parameter $\Pr(Y < X)$, when X and Y are independent and both are three-parameter Weibull distributions. Erylmaz (2010) studied stress-strength reliability for a general coherent system and illustrated the estimation procedure for exponential stress-strength distributions. Dattner (2013) considered non-parametric estimation of $\Pr(Y < X)$ when both X and Y are observed with additional errors. Recently, some authors have considered the statistical inference for R based on record data. We

2010 *Mathematics Subject Classification*. Primary 62F10; Secondary 62G30
Keywords. Stress-Strength, Record ranked set sampling, *MLE*, *UMVUE*, Bootstrap confidence interval.
Received: 15 November 2013; Accepted: 18 March 2014
Communicated by Miroslav Ristić
*Corresponding author.
Email addresses: salehi2sms@gmail.com (Mahdi Salehi), ahmadi-j@um.ac.ir (Jafar Ahmadi)

recall that there are some situations such as in destructive stress testing, the experiments have been done sequentially and only record-breaking data are observed. An example of such a set-up is the destructive testing of wooden beams in which the first beam is subjected to increasing stress until it breaks; thereafter beams are subjected to increasing stress until they break or the stress reaches the stress needed to break the previous broken beam. In this way a beam breaks only if its strength is a lower record value; see Glick (1978), Ahmadi and Arghami (2003) and Gulati and Padgett (2003). In record set-up, this scheme is known as *inverse sampling plan*. For formal definition of records, let $\{X_i,\ i \geq 1\}$ be a sequence of independent and identically distributed (iid) continuous random variables. Then, an observation X_j is called an *upper record* value if its value exceeds all previous observations, i.e., X_j is an upper record if $X_j > X_i$ for every $i < j$. These type of data are of interest and importance in several applications such as meteorological analysis, sporting and athletic events, reliability analysis specially in studying minimal repair policy and non-homogeneous Poisson process. We refer the reader to Arnold et al. (1998) for more details on record values. Among some works about stress-strength reliability based on records, Baklizi (2008a and 2014) studied point and interval estimation of the stress-strength reliability using record data in the one and two-parameter exponential distributions. Baklizi (2008b) considered the likelihood and Bayesian estimation of stress-strength reliability using lower record values from the generalized exponential distribution.

Mutllak et al. (2010) considered estimation of R using ranked set sampling (*RSS*) in the case of exponential distribution. We recall that the *RSS* is a sampling procedure that can be used to improve the cost efficiency of selecting sample units of an experiment and can be viewed as a generalization of the simple random sampling (*SRS*). It is recommended when the process of measuring sample units could be easily ranked than measured. We refer the reader to Chen et al. (2004) for pertinent details on theory and applications of ranked set sampling. This sampling motivate us to study the estimation of the stress-strength reliability based on a new sampling scheme in record-breaking data. More specifically, suppose n independent sequences are considered sequentially, the ith sequence sampling is terminated when the ith record is observed. The only observations available for analysis are the last record value in each sequences. Let us denote the last record for the ith sequence in this plan by $R_{i,i}$, then the available observations are $\mathbf{R} = (R_{1,1}, R_{2,2}, ..., R_{n,n})^T$, i.e.

$$
\begin{array}{lllllcl}
1: & \underline{R_{(1)1}} & & & & \to & R_{1,1} = R_{(1)1} \\
2: & R_{(1)2} & \underline{R_{(2)2}} & & & \to & R_{2,2} = R_{(2)2} \\
\vdots & \vdots & \vdots & \ddots & & \vdots & \vdots \\
n: & R_{(1)n} & R_{(2)n} & \ldots & \underline{R_{(n)n}} & \to & R_{n,n} = R_{(n)n}
\end{array}
$$

where $R_{(i)j}$ is the ith ordinary (upper) record in the jth sequence. Notice that unlike the ordinary records, here $R_{i,i}$'s are independent random variables but not ordered. This scheme proposed by Salehi and Ahmadi (2014). In fact, the proposed scheme is based on general *RSS*, so, we call this design *record ranked set sampling* (*RRSS*). Let $F(.;\theta)$ and $f(.;\theta)$ be the cumulative distribution function (cdf) and probability density function (pdf) of the sampling population, respectively. Then, by using the marginal density of ordinary record (see, Arnold et al., 1998) the joint density of $\mathbf{R}$ is readily obtained as

$$
f_{\mathbf{R}}(\mathbf{r};\theta) = \prod_{i=1}^{n} \frac{\{-\log(1-F(r_{i,i};\theta))\}^{i-1}}{(i-1)!} f(r_{i,i};\theta), \quad \theta \in \Theta, \tag{1}
$$

where $\mathbf{r} = (r_{1,1}, r_{2,2}, ..., r_{n,n})^T$ is the observed vector of $\mathbf{R}$, θ is real-valued parameter and Θ is the parameter space.

As an example for proposed plan, consider a parallel repairable system with *minimal repairs*, consisting of n identical components work independently with common cdf F. It is to be noted that minimal repair means that the system is brought to the condition it had immediately before the failure occurred, i.e. the age of the system is not changed as a result of the repair. Let us assume that the ith component ($i = 1, ..., n$) can be repaired $(i-1)$ times, i.e, it is not repairable after the ith its failure. Hence, the $\frac{n(n+1)}{2}$th failure is fatal to the

system and the lifetime of the system is given by $\max\{T_1, ..., T_n\}$, where T_i is the lifetime of the ith component. On the other hand, minimal repair process has the same distribution as the process of upper record values derived from iid observations with distribution F, see for example, Brown and Proschan (1983), Ahmadi and Arghami (2001) and Balakrishnan et al. (2009). Consequently, T_i is identical in distribution with $R_{i,i}$, in proposed plan. While system's lifetime is calculated according to $\max\{R_{1,1}, ..., R_{n,n}\}$, it will be adequate to know each $R_{i,i}$ to acquire the whole system's lifetime.
The rest of this paper is organized as follows: In Section 2, we derive the *MLE* and *UMVUE* of R as well as their statistical proprieties and then compare them on the basis of mean squared error (*MSE*). Some confidence intervals for R are derived and compared in Sections 3 and 4. An illustrative example is considered in Section 5.

2. Point Estimation

Let us recall that a random variable Z is said to have an exponential distribution with mean $\theta(> 0)$ denoted by $Z \sim Exp(\theta)$, if its pdf and cdf, respectively, are

$$f(z;\theta) = \frac{1}{\theta} e^{-\frac{z}{\theta}} \quad \text{and} \quad F(z;\theta) = 1 - e^{-\frac{z}{\theta}}, \quad z > 0, \ \theta > 0. \tag{2}$$

The exponential distribution is the simplest and most important distribution in reliability studies, and is applied in a wide variety of statistical procedures, especially in life testing problems. See, for example, Balakrishnan and Basu (1995). Let X and Y be two independent random variables following one-parameter exponential distributions with the parameters θ_1 and θ_2, respectively. Also, suppose $R = \Pr(X < Y)$ is the stress-strength reliability. It is easy to see that in this case $R = \dfrac{\theta_2}{\theta_2 + \theta_1}$. We are interested in estimating the stress-strength R when the samples are permitted to be upper *RRSS's* with possibly different number of observations. More specifically, let $\mathbf{r} = (r_{1,1}, r_{2,2}, ..., r_{n,n})^T$ be the observation of random vector $\mathbf{R} = (R_{1,1}, R_{2,2}, ..., R_{n,n})^T$, an upper *RRSS* of size n from $Exp(\theta_1)$, and $\mathbf{s} = (s_{1,1}, s_{2,2}, ..., s_{m,m})^T$ be the observation of the random vector $\mathbf{S} = (S_{1,1}, S_{2,2}, ..., S_{m,m})^T$, an upper *RRSS* of size m from $Exp(\theta_2)$.

2.1. MLE

First, we find the MLE of θ_1 and θ_2. By substituting (2) into (1), the likelihood functions follow as

$$L_1(\theta_1; \mathbf{r}) = \frac{\theta_1^{-N}}{\prod_{i=1}^{n}(i-1)!} \exp\left(-\frac{1}{\theta_1}\sum_{i=1}^{n} r_{i,i}\right), \tag{3}$$

$$L_2(\theta_2; \mathbf{s}) = \frac{\theta_2^{-M}}{\prod_{j=1}^{m}(j-1)!} \exp\left(-\frac{1}{\theta_2}\sum_{j=1}^{m} s_{j,j}\right), \tag{4}$$

where $N = \frac{n(n+1)}{2}$ and $M = \frac{m(m+1)}{2}$. Then, the MLE of the parameters θ_1 and θ_2 can be readily given by

$$\hat{\theta}_{1(ML)} = \frac{1}{N}\sum_{i=1}^{n} R_{i,i} \quad \text{and} \quad \hat{\theta}_{2(ML)} = \frac{1}{M}\sum_{j=1}^{m} S_{j,j}, \tag{5}$$

respectively. Using the invariance property of the *MLE*, we find the *MLE* of R as

$$\hat{R}_{ML} = \frac{\hat{\theta}_{2(ML)}}{\hat{\theta}_{2(ML)} + \hat{\theta}_{1(ML)}} = \left(1 + \frac{M}{N}\frac{\sum_{i=1}^{n} R_{i,i}}{\sum_{j=1}^{m} S_{j,j}}\right)^{-1}. \tag{6}$$

From Arnold et al. (1998, pp. 20), $R_{i,i}$ has Gamma-distribution with parameters i and θ_1, denoted by $R_{i,i} \sim Gamma(i, \theta_1)$ with pdf

$$f_{R_{i,i}}(r_{i,i}; \theta_1) = \frac{r_{i,i}^{i-1}}{\Gamma(i)\theta_1} e^{-\frac{r_{i,i}}{\theta_1}}, \quad r_{i,i} > 0,$$

where $\Gamma(.)$ is the complete gamma function. As it is mentioned earlier, $R_{i,i}$'s are independent random variables. Therefore, $\sum_{i=1}^{n} R_{i,i} \sim Gamma(N, \theta_1)$ and similarly $\sum_{j=1}^{m} S_{j,j} \sim Gamma(M, \theta_2)$. Consequently, from (6) we have

$$\hat{R}_{ML} \overset{d}{=} \left(1 + \frac{1-R}{R} F_{2N,2M}\right)^{-1}, \tag{7}$$

where $\overset{d}{=}$ means identical in distribution and $F_{2N,2M}$ stands for the F-distribution with $2N$ and $2M$ degrees

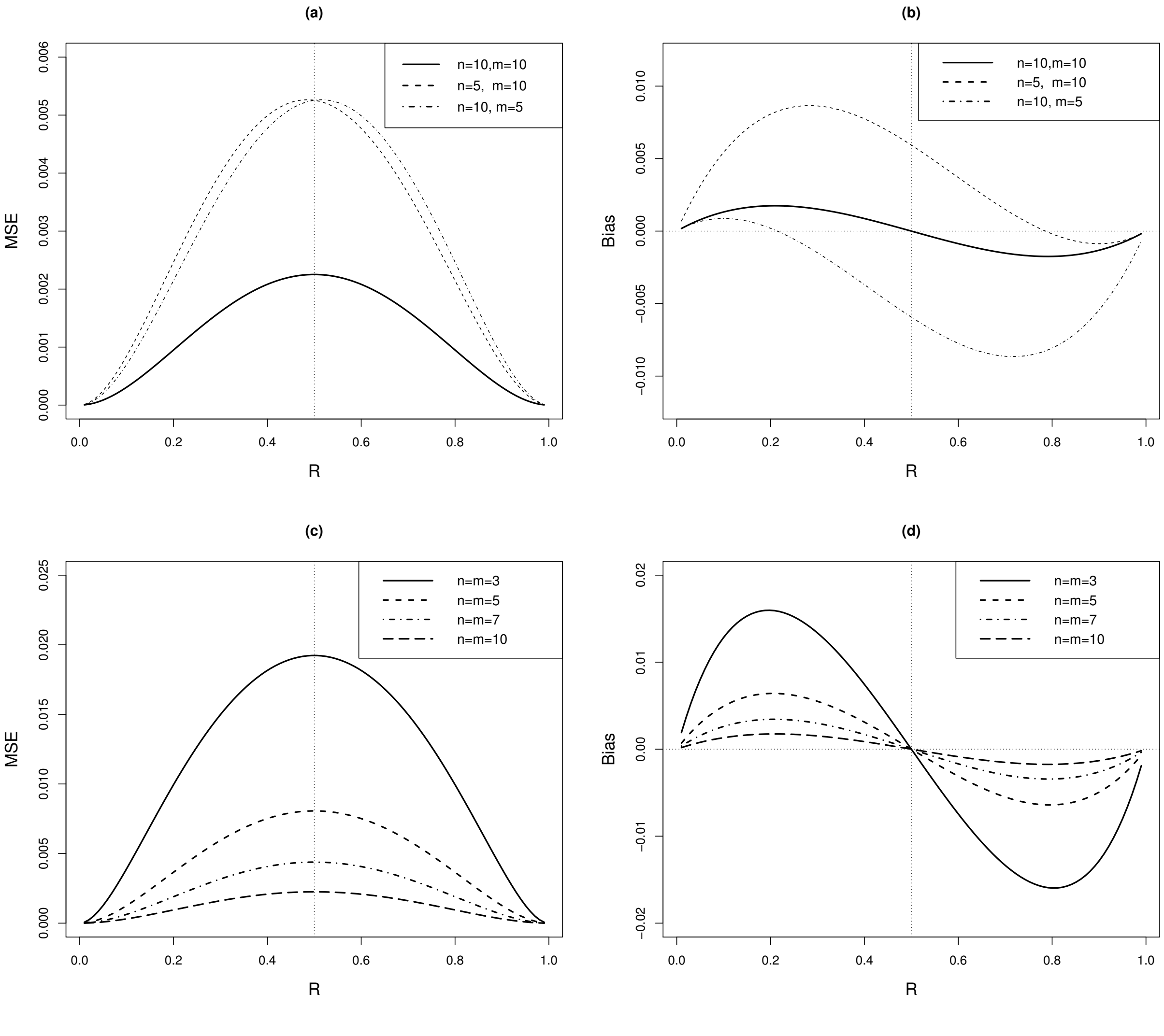

Figure 1: Plot of $MSE(\hat{R}_{ML}, R)$ and $Bias(\hat{R}_{ML}, R)$ versus R.

of freedoms. We use (7) for obtaining the bias and MSE of $\hat{R}_{ML}$, i.e. $Bias(\hat{R}_{ML}, R) = E\left(\hat{R}_{ML} - R\right)$ and

$MSE(\hat{R}_{ML}, R) = E\left(\hat{R}_{ML} - R\right)^2$, respectively. It is easy to see that by substituting $R = 0.5$ into (7) yields $\hat{R}_{ML} \overset{d}{=} (1 + F_{2N,2M})^{-1}$, and then one can show that $E\left(\hat{R}_{ML}\right) = 0.5$ when $n = m$. In Figures 1 and 2, we plot the numerical values of *Bias* and *MSE* of $\hat{R}_{ML}$ versus R, for some selected values of n and m. From these

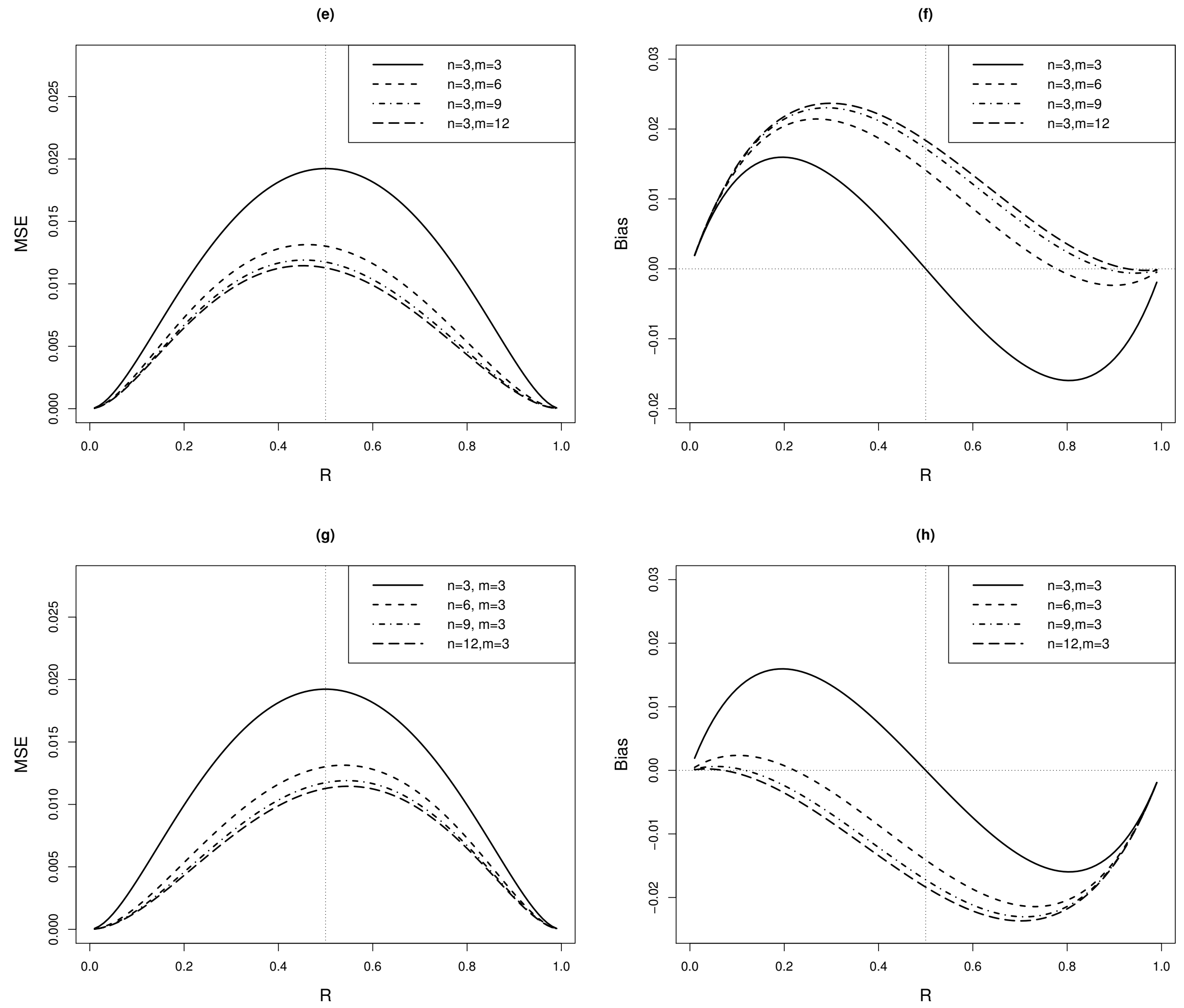

Figure 2: Continued.

figures, we observe the following points:

- As it is expected, the *MSE* and the *Bias* are reduced by increasing the sample sizes [see, Figure 1, parts (c) and (d)].
- When $n < m$, the performance of the *MLE* is better for $R = 0.5 + \gamma$ comparing to $R = 0.5 - \gamma$, where $0 < \gamma < 0.5$. [see, Figure 2, parts (e) and (f)].
- Let $MSE(\hat{R}_{ML}, R_{\max}) = \max_{R} MSE(\hat{R}_{ML}, R)$, then we observe that $R_{\max} \propto \frac{n}{n+m}$ [see, Figure 2, parts (e) and (g)].

- If $n < m$, we have over-estimation and else we have under-estimation and also $Bias(\hat{R}_{ML}, R)$ is symmetric, say about the point $(0.5, 0)$, when $n = m$ [see, Figure 1, part (d) and Figure 2, parts (f) and (h)].
- The $MSE(\hat{R}_{ML}, R)$ is symmetric, say about $R = 0.5$, when $n = m$, and departures from symmetry when $n < m$ and $n > m$, respectively [see, Figure 1, part (a)].
- As mentioned earlier, $\hat{R}_{ML}$ is unbiased when $R = 0.5$ and $m = n$, this is confirmed by Figure 1 part (d).

2.2. UMVUE

It should be mentioned that when $n = m = 1$, the *UMVUE* of R does not exist [see, Kotz et al. (2003)]. So, we consider the following three cases:

(i) **Case 1:** $\min\{m, n\} \geq 2$:

Since $R_{1,1} \stackrel{d}{=} X$ and $S_{1,1} \stackrel{d}{=} Y$, so $I(R_{1,1} < S_{1,1})$ is an unbiased estimator of R, where $I(A) = 1$, if the event A occurs and $I(A) = 0$, otherwise. Also, from (3) and (4) it could be seen that $\left(\sum_{i=1}^{n} R_{i,i}, \sum_{j=1}^{m} S_{j,j}\right)^T$ is a complete sufficient statistic for (θ_1, θ_2). Thus, by applying the Rao-Blackwell and Lehmann-Scheffe's Theorem [see, e.g. Lehmann and Cassela (1998)], gives the *UMVUE* of R as

$$\begin{aligned} \hat{R}_{UMVU} &= \Pr\left\{R_{1,1} < S_{1,1} \mid \sum_{i=1}^{n} R_{i,i}, \sum_{j=1}^{m} S_{j,j}\right\} \\ &= \Pr\left\{W < \frac{\sum_{j=1}^{m} S_{j,j}}{\sum_{i=1}^{n} R_{i,i}} \mid \sum_{i=1}^{n} R_{i,i}, \sum_{j=1}^{m} S_{j,j}\right\}, \end{aligned} \tag{8}$$

where $W = \dfrac{R_{1,1}}{S_{1,1}} \dfrac{\sum_{j=1}^{m} S_{j,j}}{\sum_{i=1}^{n} R_{i,i}} \stackrel{d}{=} \dfrac{Beta(1, N-1)}{Beta(1, M-1)}$ is an ancillary statistic. Then, by Basu's Theorem, it is independent of the complete sufficient statistic. So we have $\hat{R}_{UMVU} = F_W\left(\sum_{j=1}^{m} S_{j,j} / \sum_{i=1}^{n} R_{i,i}\right)$, where $F_W(.)$ is the cdf of the random variable W. Finally, by doing some algebraic manipulation, we obtain

$$\hat{R}_{UMVU} = \begin{cases} 1 - Q\left(\hat{R}_{ML}; n, m\right), & if \quad \hat{R}_{ML} \leq \frac{N}{N+M}, \\ Q\left(1 - \hat{R}_{ML}; m, n\right), & if \quad \hat{R}_{ML} \geq \frac{N}{N+M}, \end{cases} \tag{9}$$

where $\hat{R}_{ML}$ is given by (6) and

$$Q(t; n, m) = \sum_{d=0}^{N-1} \frac{\binom{N-1}{d}}{\binom{d+M-1}{d}} \left(-\frac{M}{N} \frac{t}{1-t}\right)^d, \tag{10}$$

with $N = n(n+1)/2$ and $M = m(m+1)/2$.

(ii) **Case 2:** $n = 1$ **and** $m \geq 2$:

In this case, obviously $\left(R_{1,1}, \sum_{j=1}^{m} S_{j,j}\right)^T$ is a complete sufficient statistic and hence by using the Rao-Blackwell and Lehmann-Scheffe's Theorem, the *UMVUE* of R is derived as follows

$$\left(1 - \frac{R_{1,1}}{\sum_{j=1}^{m} S_{j,j}}\right)^{M-1} I\left(R_{1,1} \leq \sum_{j=1}^{m} S_{j,j}\right). \tag{11}$$

(iii) **Case 3:** $m = 1$ **and** $n \geq 2$:

Similar to the **Case 2**, one can show that the *UMVUE* of R is

$$1 - \left(1 - \frac{S_{1,1}}{\sum_{i=1}^{n} R_{i,i}}\right)^{N-1} I\left(S_{1,1} \leq \sum_{i=1}^{n} R_{i,i}\right). \tag{12}$$

From (9), (10) and using (7), we have plotted the numerical values of $Var\left(\hat{R}_{UMVU}\right)$ versus R, for some choices of n and m in Figure 3. From this figure, it is observed that variance of $\hat{R}_{UMVU}$ is symmetric, say about

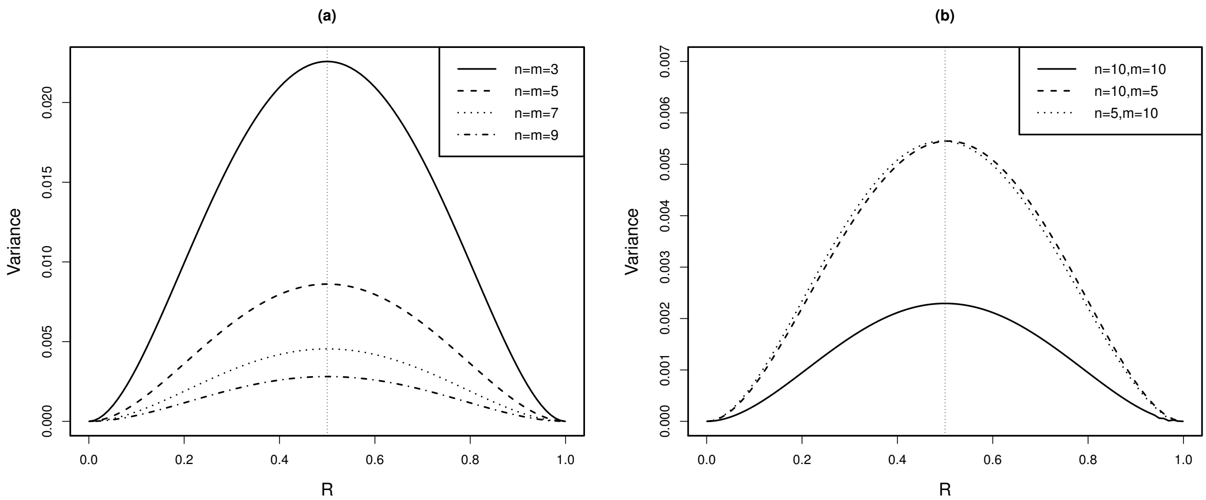

Figure 3: Plot of $Var(\hat{R}_{UMVU}, R)$ versus R.

$R = 0.5$, when $n = m$ and is decreasing when the sample sizes increase (as we expected). It is observed from Figure 3 that the behaviour of the variance of $\hat{R}_{UMVU}$ is almost similar to the *MSE* of $\hat{R}_{ML}$.

2.3. Comparison

In this section, we intend to compare $\hat{R}_{UMVU}$ and $\hat{R}_{ML}$, as $\hat{R}_{UMVU}$ is unbiased, so we consider the *MSE* as a criterion. To this end, we have plotted *MSE* and variance of $\hat{R}_{ML}$ and $\hat{R}_{UMVU}$, respectively, in Figure 4 for some selected values of m and n. From this figure, we observe that *MSE* of $\hat{R}_{ML}$ is less then the variance of $\hat{R}_{UMVU}$ for the values of R near to 0.5 specially for the small sample sizes.

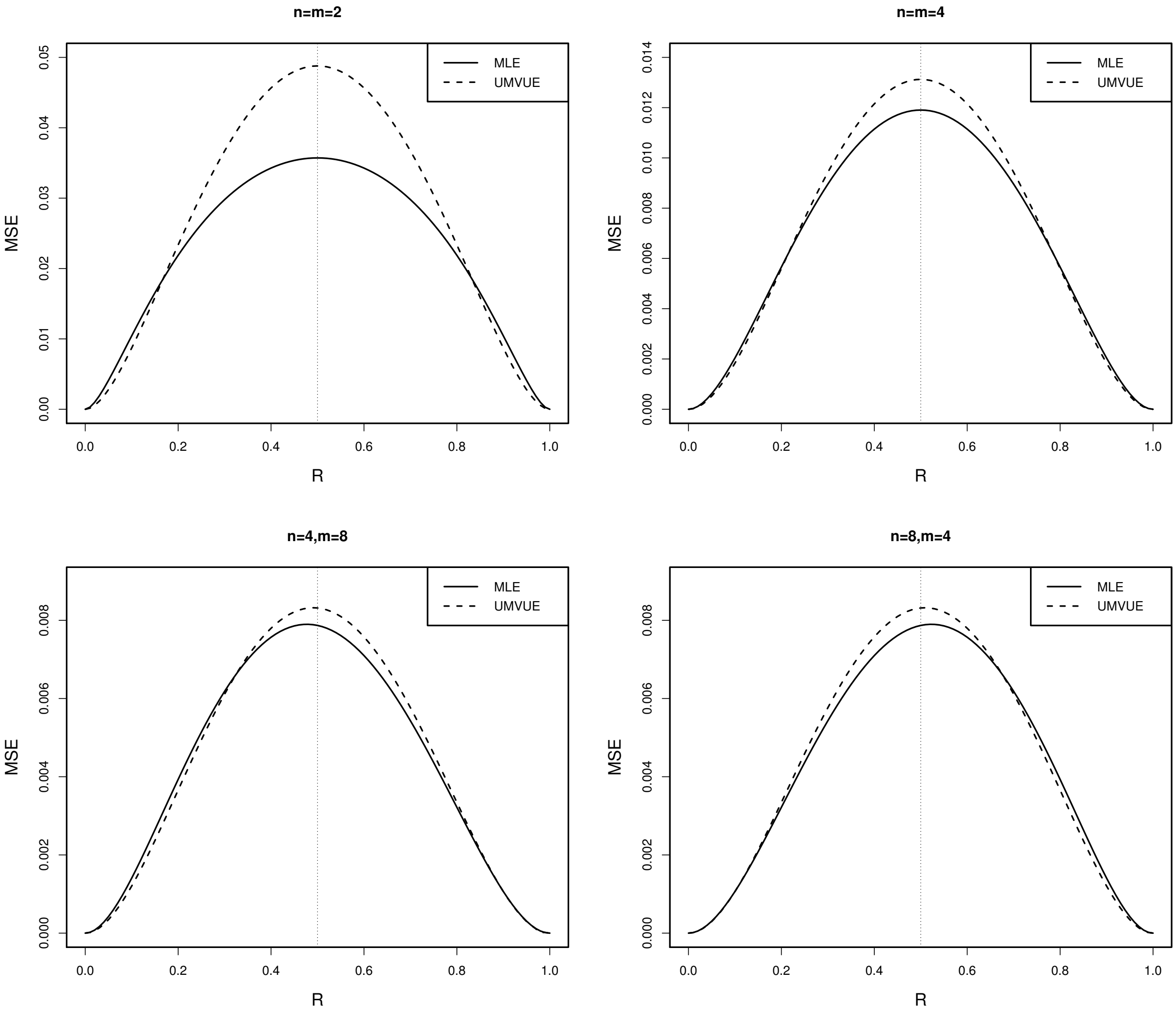

Figure 4: Comparison of $MSE(\hat{R}_{ML}, R)$ and $Var(\hat{R}_{UMVU}, R)$ for some selected values of n and m.

3. Interval Estimation

We consider three methods to construct confidence interval (CI) for R.

3.1. CI Based on a Pivotal Quantity

From (7), a $100(1-\alpha)\%$ CI for R is derived as follows

$$\Pr\left\{\left(1+\frac{1-\hat{R}_{ML}}{\hat{R}_{ML}}F_{2M,2N}\left(1-\frac{\alpha}{2}\right)\right)^{-1} \leq R \leq \left(1+\frac{1-\hat{R}_{ML}}{\hat{R}_{ML}}F_{2M,2N}\left(\frac{\alpha}{2}\right)\right)^{-1}\right\} = 1-\alpha, \tag{13}$$

where $F_{2M,2N}(\gamma)$ is the γth quantile of $F_{2M,2N}$-distribution. We can use (7) to obtain the expected length of CI in (13).

3.2. Approximate CI

Consider a situation that the number of record values is sufficiently large. Hence we may use the asymptotic confidence interval based on the limit distribution of $\hat{R}_{ML}$. Let us denote the convergence in distribution by $\xrightarrow{d}$. We know that as $n, m \to \infty$, then $\left(\hat{\theta}_1 - \theta_1\right) \xrightarrow{d} N(0, \sigma_1^2)$ and $\left(\hat{\theta}_2 - \theta_2\right) \xrightarrow{d} N(0, \sigma_2^2)$, where $\sigma_1^2 = -1/E\left(\frac{\partial^2}{\partial\theta_1^2}\log L_1(\theta_1; \mathbf{R})\right)$ and $\sigma_2^2 = -1/E\left(\frac{\partial^2}{\partial\theta_2^2}\log L_2(\theta_2; \mathbf{S})\right)$ with L_1 and L_2 given in (3) and (4), respectively. It is easy to show that $\sigma_1^2 = \frac{\theta_1^2}{N}$ and $\sigma_2^2 = \frac{\theta_2^2}{M}$. Now, suppose that n, m are sufficiently large in such a way that $\frac{M}{N} \to p$, where $p \in (0, 1)$, so we readily conclude that

$$\sqrt{N}\begin{pmatrix} \hat{\theta}_{1(ML)} - \theta_1 \\ \hat{\theta}_{2(ML)} - \theta_2 \end{pmatrix} \xrightarrow{d} N_2\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \theta_1^2 & 0 \\ 0 & \frac{\theta_2^2}{p} \end{pmatrix}\right).$$

Hence, taking $g(t_1, t_2) = \frac{t_2}{t_1+t_2}$ and then a simple application of the multivariate delta method yields [see, Wasserman (2006, pp. 5-6)]

$$\left(\hat{R}_{ML} - R\right) = \left(g\left(\hat{\theta}_{1(ML)}, \hat{\theta}_{2(ML)}\right) - g\left(\theta_1, \theta_2\right)\right) \xrightarrow{d} N(0, \omega^2), \tag{14}$$

where

$$\omega = \frac{\theta_1\theta_2}{(\theta_1 + \theta_2)^2}\sqrt{\frac{1}{N}\left(1 + \frac{1}{p}\right)}. \tag{15}$$

By using the property in (14) and applying the Slutsky's Theorem, a $100(1-\alpha)\%$ approximate *CI* for R is derived as follows

$$\left(\hat{R}_{ML} - z_{1-\frac{\alpha}{2}}\hat{\omega},\ \hat{R}_{ML} + z_{1-\frac{\alpha}{2}}\hat{\omega}\right), \tag{16}$$

where z_γ stands for the γth quantiles of the standard normal distribution and $\hat{\omega}$ is obtained by substituting the plug-in estimators $\hat{\theta}_{1(ML)}$ and $\hat{\theta}_{2(ML)}$ instead of θ_1 and θ_2, respectively, in (15). Similarly, if $n < m$ we get the approximate *CI* for R as $\left(\hat{R}_{ML} - z_{1-\frac{\alpha}{2}}\hat{\delta},\ \hat{R}_{ML} + z_{1-\frac{\alpha}{2}}\hat{\delta}\right)$, where $\hat{\delta} = \frac{\hat{\theta}_{1(ML)}\hat{\theta}_{2(ML)}}{(\hat{\theta}_{1(ML)}+\hat{\theta}_{2(ML)})^2}\sqrt{\frac{1}{M}(1+p)}$.

3.3. *Parametric Bootstrap CIs*

All inference procedures in this paper are obtained based on $\sum_{i=1}^n R_{i,i}$ and $\sum_{j=1}^m S_{j,j}$ and using the fact that $\sum\limits_{i=1}^n R_{i,i} \sim Gamma(N, \theta_1)$ and $\sum\limits_{j=1}^m S_{j,j} \sim Gamma(M, \theta_2)$, so we can use the parametric bootstrap *CI*s. There are several ways to construct bootstrap *CI*s. But, the percentile *CI* and bootstrap-*t* *CI* are commonly used for the stress-strength. For more details on the various methods to construct bootstrap *CI*s and also using R Software see Efron and Tibshirani (1993) and Rizzo (2008), respectively. The following algorithms are used to construct the parametric bootstrap *CI*s for R in this paper.

Algorithm 3.1. *(Percentile CI)*

Step 1. *Based on the independent observed samples* $\mathbf{r} = (r_{1,1}, r_{2,2}, ..., r_{n,n})^T$ *and* $\mathbf{s} = (s_{1,1}, s_{2,2}, ..., s_{m,m})^T$, *calculate* $\hat{\theta}_{1(ML)}$, $\hat{\theta}_{2(ML)}$ *and* $\hat{R}_{ML}$ *from (5) and (6), respectively.*

Step 2. *Generate* $r_{i,i}^\star \sim Gamma(i, \hat{\theta}_{1(ML)}), i = 1, ..., n$ *and* $s_{j,j}^\star \sim Gamma(j, \hat{\theta}_{2(ML)}), j = 1, ..., m$. *Use* $\mathbf{r}^\star = \left(r_{1,1}^\star, r_{2,2}^\star, ..., r_{n,n}^\star\right)^T$ *and* $\mathbf{s}^\star = \left(s_{1,1}^\star, s_{2,2}^\star, ..., s_{m,m}^\star\right)^T$ *to calculate* $\hat{\theta}_{1(ML)}^\star$, $\hat{\theta}_{2(ML)}^\star$ *and* $\hat{R}_{ML}^\star$.

Step 3. *Repeat Step 2 for* $b = 1, ..., B$, *to derive* $\hat{R}_{(b)ML}^\star$, $b = 1, ..., B$.

Step 4. *Let* $\hat{H}$ *be the empirical cumulative distribution function based on the parametric bootstrap estimates* $\hat{R}_{(b)ML}^\star$, $b = 1, ..., B$, *then the* $100(1-\alpha)\%$ *percentile CI of* R *is* $\left(\hat{H}^{-1}\left(\frac{\alpha}{2}\right),\ \hat{H}^{-1}\left(1 - \frac{\alpha}{2}\right)\right)$.

Algorithm 3.2. *(Bootstrap-t CI based on the asymptotic standard deviation ω in (15))*

Step 1. *Do Step 1 of Algorithm 3.1 and also compute $\hat{\omega}$ in (15) with $\hat{\theta}_{1(ML)}$ and $\hat{\theta}_{2(ML)}$ instead of θ_1 and θ_2, respectively.*

Step 2. *Do Step 2 of Algorithm 3.1 and also compute $\hat{\omega}^{\star}$ given in (15) substituting $\hat{\theta}^{\star}_{1(ML)}$ and $\hat{\theta}^{\star}_{2(ML)}$ instead of θ_1 and θ_2, respectively.*

Step 3. *Repeat Step 2 for $b = 1, ..., B$, to derive $\hat{R}^{\star}_{(b)ML}$ and $\hat{\omega}^{\star}_{(b)}$, $b = 1, ..., B$.*

Step 4. *Let $\mathbf{Z}^{\star} = \left(z^{\star}_{(1)}, ..., z^{\star}_{(B)}\right)^T$, where $z^{\star}_{(b)} = \frac{\hat{R}^{\star}_{(b)ML} - \hat{R}}{\hat{\omega}^{\star}_{(b)}}$, $b = 1, ..., B$.*

Step 5. *Compute the $100(1-\alpha)\%$ bootstrap-t CI for R as $\left(\hat{R} - z^{\star}_{1-\frac{\alpha}{2}}\hat{\omega},\ \hat{R} - z^{\star}_{\frac{\alpha}{2}}\hat{\omega}\right)$, where $z^{\star}_{\gamma}$ is the γth quantile of $\mathbf{Z}^{\star}$ given by Step 4.*

Algorithm 3.3. *(Bootstrap-t CI based on the bootstrap variance estimate)*

Step 1. *Do Steps 1-2 of Algorithm 3.1.*

Step 2. *Do Step 3 of Algorithm 3.1.*

Step 3. *For each iteration of Step 2, design another parametric bootstrap procedure to estimate the standard deviation of $\hat{R}^{\star}_{(b)ML}$, say $\hat{\sigma}\left(\hat{R}^{\star}_{(b)ML}\right)$. More precisely, repeat Step 2 of Algorithm 3.1 for $b' = 1, ..., B'$, with $\hat{\theta}^{\star}_{1(ML)}$ and $\hat{\theta}^{\star}_{2(ML)}$ instead of $\hat{\theta}_{1(ML)}$ and $\hat{\theta}_{2(ML)}$, respectively, and then calculate*

$$\hat{\sigma}\left(\hat{R}^{\star}_{(b)ML}\right) = \sqrt{\frac{1}{B'-1}\sum_{b'=1}^{B'}\left(\hat{R}^{\star\star}_{(b')ML} - \bar{R}^{\star\star}\right)^2},$$

where $\bar{R}^{\star\star} = \frac{1}{B'}\sum\limits_{b'=1}^{B'}\hat{R}^{\star\star}_{(b')ML}$.

Step 4. *Let $\mathbf{t}^{\star} = \left(t^{\star}_{(1)}, ..., t^{\star}_{(B)}\right)^T$, where $t^{\star}_{(b)} = \frac{\hat{R}^{\star}_{(b)ML} - \hat{R}}{\hat{\sigma}\left(\hat{R}^{\star}_{(b)ML}\right)}$, $b = 1, ..., B$.*

Step 5. *Compute the $100(1-\alpha)\%$ bootstrap-t CI for R as $\left(\hat{R} - t^{\star}_{1-\frac{\alpha}{2}}\hat{\omega},\ \hat{R} - t^{\star}_{\frac{\alpha}{2}}\hat{\omega}\right)$, where $t^{\star}_{\gamma}$ is the γth quantile of $\mathbf{t}^{\star}$ given by Step 4.*

4. Simulation Study

In the present section, we consider a simulation study for comparing the *CI*s obtained in the previous section. All combination of $n = 3, 5, 7$, $m = 3, 5, 7$, $\theta_1 = 1$, $R = 0.1, 0.3, 0.5, 0.95, 0.99$ and $\alpha = 0.05, 0.1$ are used. In each combination, 1000 samples of $\mathbf{r}$ and $\mathbf{s}$ from $Exp(\theta_1)$ and $Exp(\theta_2)$ are simulated, respectively. Notice that, we fix $B = 200$ and $B' = 25$. We generated the following *CI*s and collected the results in Tables 1 and 2.

- *Perc*: The parametric percentile *CI* obtained based on Algorithm 3.1.
- *Boot*$_1$-*t*: The parametric bootstrap-*t CI* obtained based on Algorithm 3.2.
- *Boot*$_2$-*t*: The parametric bootstrap-*t CI* obtained based on Algorithm 3.3.
- *MLE*: The *CI* based on pivotal quantity given by (13).
- *AMLE*: The asymptotic *CI* given in (16).

Table 1: The coverage probability (*C.P.*) and expected length (*E.L.*) of R for α=0.05.

			Perc		$Boot_1$-t		$Boot_2$-t		*MLE*		*AMLE*	
R	n	m	*C.P.*	*E.L.*	*C.P.*	*E.L.*	*C.P.*	*E.L.*	*C.P.*	*E.L.*	*C.P.*	*E.L.*
0.10	3	3	0.943	0.246	0.973	0.297	0.965	0.281	0.947	0.245	0.908	0.218
0.10	3	5	0.934	0.215	0.976	0.210	0.984	0.239	0.949	0.187	0.966	0.221
0.10	3	7	0.921	0.207	0.971	0.191	0.989	0.253	0.951	0.172	0.988	0.240
0.10	5	3	0.937	0.174	0.968	0.230	0.986	0.284	0.945	0.201	0.931	0.204
0.10	5	5	0.952	0.139	0.962	0.149	0.957	0.151	0.955	0.139	0.936	0.132
0.10	5	7	0.937	0.126	0.950	0.127	0.978	0.143	0.944	0.121	0.959	0.133
0.10	7	3	0.939	0.156	0.967	0.215	0.995	0.325	0.961	0.190	0.961	0.216
0.10	7	5	0.943	0.116	0.948	0.129	0.979	0.149	0.950	0.122	0.951	0.128
0.10	7	7	0.950	0.099	0.955	0.103	0.965	0.106	0.960	0.099	0.947	0.096
0.30	3	3	0.948	0.451	0.982	0.609	0.973	0.607	0.953	0.452	0.901	0.458
0.30	3	5	0.940	0.397	0.974	0.460	0.985	0.558	0.949	0.375	0.959	0.473
0.30	3	7	0.934	0.380	0.968	0.422	0.991	0.593	0.945	0.352	0.980	0.531
0.30	5	3	0.940	0.362	0.974	0.489	0.992	0.616	0.956	0.387	0.945	0.455
0.30	5	5	0.945	0.292	0.963	0.329	0.967	0.341	0.949	0.293	0.926	0.294
0.30	5	7	0.942	0.263	0.960	0.281	0.977	0.328	0.952	0.258	0.960	0.293
0.30	7	3	0.944	0.342	0.970	0.460	0.997	0.700	0.954	0.372	0.972	0.515
0.30	7	5	0.960	0.255	0.977	0.287	0.993	0.339	0.967	0.261	0.968	0.291
0.30	7	7	0.953	0.216	0.965	0.230	0.969	0.241	0.960	0.216	0.941	0.217
0.50	3	3	0.957	0.502	0.975	0.689	0.970	0.695	0.961	0.502	0.899	0.523
0.50	3	5	0.935	0.431	0.971	0.541	0.985	0.679	0.949	0.434	0.955	0.542
0.50	3	7	0.910	0.407	0.969	0.502	0.996	0.744	0.951	0.410	0.970	0.605
0.50	5	3	0.946	0.431	0.975	0.552	0.982	0.690	0.945	0.435	0.951	0.543
0.50	5	5	0.935	0.338	0.956	0.383	0.960	0.402	0.934	0.339	0.913	0.346
0.50	5	7	0.944	0.301	0.970	0.332	0.987	0.393	0.954	0.301	0.958	0.343
0.50	7	3	0.932	0.410	0.984	0.516	0.994	0.749	0.947	0.411	0.967	0.609
0.50	7	5	0.945	0.301	0.966	0.333	0.988	0.395	0.948	0.301	0.957	0.343
0.50	7	7	0.945	0.255	0.965	0.273	0.970	0.288	0.952	0.254	0.933	0.257
0.95	3	3	0.934	0.149	0.966	0.161	0.952	0.149	0.941	0.148	0.899	0.122
0.95	3	5	0.919	0.098	0.941	0.122	0.970	0.150	0.933	0.116	0.905	0.109
0.95	3	7	0.913	0.086	0.952	0.112	0.983	0.168	0.948	0.108	0.935	0.113
0.95	5	3	0.920	0.127	0.957	0.112	0.963	0.123	0.942	0.107	0.950	0.120
0.95	5	5	0.939	0.077	0.950	0.078	0.952	0.078	0.948	0.077	0.917	0.071
0.95	5	7	0.948	0.063	0.962	0.067	0.979	0.078	0.959	0.067	0.955	0.069
0.95	7	3	0.920	0.120	0.951	0.100	0.980	0.129	0.945	0.096	0.979	0.127
0.95	7	5	0.941	0.069	0.955	0.066	0.971	0.073	0.945	0.065	0.958	0.071
0.95	7	7	0.941	0.053	0.939	0.053	0.948	0.054	0.945	0.053	0.936	0.051
0.99	3	3	0.948	0.033	0.964	0.033	0.951	0.029	0.952	0.033	0.908	0.025
0.99	3	5	0.952	0.022	0.955	0.026	0.984	0.032	0.958	0.027	0.940	0.023
0.99	3	7	0.925	0.019	0.946	0.024	0.987	0.036	0.945	0.024	0.939	0.023
0.99	5	3	0.936	0.030	0.949	0.024	0.968	0.025	0.951	0.024	0.958	0.025
0.99	5	5	0.941	0.017	0.947	0.017	0.946	0.016	0.951	0.017	0.938	0.015
0.99	5	7	0.941	0.013	0.948	0.014	0.969	0.016	0.953	0.014	0.940	0.014
0.99	7	3	0.928	0.028	0.956	0.021	0.984	0.027	0.953	0.022	0.983	0.027
0.99	7	5	0.936	0.015	0.942	0.014	0.963	0.015	0.943	0.014	0.959	0.015
0.99	7	7	0.949	0.011	0.951	0.011	0.951	0.011	0.951	0.011	0.932	0.011

From Tables 1 and 2 we observe the following points:

- In the all of *CI*s we see that the expected length is almost decreasing when the sample sizes increase (as we expect).
- It seems that the maximum of the expected length occurs at $R = 0.5$ and the expected lengths are very small for the extreme values of R, namely for 0.95 and 0.99 (similar to the *MSE* of the point estimators).
- The percentile *CI* is better than the other bootstrap *CI*s since its expected length is smaller specially for the values of R close to 0.5.
- The $Boot_1$-t *CI* works well compared to the $Boot_2$-t *CI* specially in the large sample sizes.

Table 2: The values of *C.P.* and *E.L.* of R for α=0.1.

			Perc		$Boot_1$-t		$Boot_2$-t		*MLE*		*AMLE*	
R	n	m	*C.P.*	*E.L.*	*C.P.*	*E.L.*	*C.P.*	*E.L.*	*C.P.*	*E.L.*	*C.P.*	*E.L.*
0.10	3	3	0.885	0.201	0.923	0.229	0.920	0.215	0.896	0.201	0.859	0.185
0.10	3	5	0.885	0.177	0.924	0.172	0.950	0.193	0.900	0.158	0.938	0.190
0.10	3	7	0.853	0.170	0.927	0.157	0.971	0.210	0.898	0.147	0.971	0.215
0.10	5	3	0.909	0.146	0.931	0.180	0.973	0.222	0.918	0.165	0.910	0.176
0.10	5	5	0.897	0.116	0.915	0.121	0.916	0.121	0.904	0.116	0.890	0.111
0.10	5	7	0.902	0.103	0.912	0.103	0.942	0.114	0.902	0.100	0.925	0.110
0.10	7	3	0.866	0.133	0.904	0.169	0.976	0.255	0.897	0.156	0.927	0.196
0.10	7	5	0.893	0.097	0.895	0.104	0.933	0.120	0.897	0.101	0.915	0.108
0.10	7	7	0.908	0.082	0.916	0.084	0.932	0.085	0.919	0.082	0.904	0.080
0.30	3	3	0.879	0.379	0.941	0.467	0.922	0.465	0.899	0.381	0.847	0.381
0.30	3	5	0.870	0.336	0.922	0.372	0.954	0.453	0.885	0.322	0.904	0.401
0.30	3	7	0.905	0.323	0.941	0.346	0.987	0.502	0.921	0.304	0.974	0.453
0.30	5	3	0.883	0.311	0.928	0.382	0.963	0.482	0.907	0.329	0.907	0.388
0.30	5	5	0.925	0.247	0.941	0.268	0.946	0.276	0.927	0.248	0.904	0.248
0.30	5	7	0.883	0.222	0.913	0.232	0.942	0.268	0.893	0.219	0.912	0.247
0.30	7	3	0.865	0.287	0.916	0.357	0.984	0.544	0.893	0.309	0.929	0.429
0.30	7	5	0.890	0.215	0.911	0.233	0.945	0.273	0.897	0.218	0.913	0.243
0.30	7	7	0.906	0.183	0.916	0.191	0.920	0.199	0.906	0.183	0.895	0.183
0.50	3	3	0.899	0.428	0.937	0.535	0.927	0.543	0.911	0.430	0.863	0.440
0.50	3	5	0.882	0.368	0.938	0.434	0.966	0.545	0.909	0.371	0.929	0.457
0.50	3	7	0.863	0.346	0.917	0.400	0.972	0.604	0.889	0.348	0.943	0.508
0.50	5	3	0.869	0.365	0.922	0.432	0.954	0.543	0.884	0.369	0.900	0.454
0.50	5	5	0.907	0.288	0.932	0.315	0.933	0.328	0.910	0.288	0.893	0.291
0.50	5	7	0.891	0.254	0.915	0.272	0.951	0.319	0.903	0.254	0.918	0.288
0.50	7	3	0.886	0.347	0.937	0.408	0.985	0.614	0.920	0.350	0.960	0.511
0.50	7	5	0.892	0.253	0.906	0.272	0.951	0.318	0.892	0.254	0.907	0.288
0.50	7	7	0.880	0.214	0.908	0.225	0.916	0.235	0.899	0.214	0.878	0.216
0.95	3	3	0.916	0.116	0.942	0.124	0.929	0.111	0.916	0.117	0.882	0.103
0.95	3	5	0.893	0.081	0.910	0.096	0.964	0.117	0.899	0.094	0.908	0.096
0.95	3	7	0.872	0.071	0.909	0.088	0.983	0.132	0.901	0.086	0.910	0.102
0.95	5	3	0.881	0.100	0.911	0.090	0.943	0.098	0.901	0.087	0.932	0.103
0.95	5	5	0.904	0.063	0.907	0.064	0.904	0.063	0.902	0.063	0.893	0.060
0.95	5	7	0.906	0.053	0.908	0.056	0.937	0.063	0.906	0.055	0.923	0.058
0.95	7	3	0.865	0.097	0.922	0.084	0.976	0.107	0.906	0.082	0.973	0.117
0.95	7	5	0.893	0.056	0.894	0.054	0.923	0.059	0.897	0.053	0.920	0.059
0.95	7	7	0.897	0.044	0.902	0.044	0.904	0.045	0.899	0.045	0.896	0.043
0.99	3	3	0.900	0.027	0.907	0.027	0.858	0.023	0.899	0.027	0.865	0.022
0.99	3	5	0.884	0.018	0.893	0.021	0.950	0.024	0.891	0.021	0.891	0.020
0.99	3	7	0.865	0.015	0.887	0.019	0.968	0.027	0.892	0.019	0.912	0.021
0.99	5	3	0.876	0.023	0.897	0.020	0.916	0.020	0.895	0.020	0.933	0.023
0.99	5	5	0.886	0.014	0.883	0.014	0.862	0.013	0.889	0.014	0.892	0.013
0.99	5	7	0.870	0.011	0.873	0.011	0.909	0.013	0.879	0.012	0.891	0.012
0.99	7	3	0.875	0.022	0.895	0.018	0.955	0.021	0.888	0.018	0.964	0.025
0.99	7	5	0.886	0.012	0.887	0.012	0.925	0.013	0.895	0.012	0.928	0.013
0.99	7	7	0.882	0.009	0.889	0.009	0.885	0.009	0.891	0.009	0.881	0.009

- As expected from the intuition, the exact *CI* of *MLE* is the best *CI* with respect to approximated *CI*s while the performance of the percentile *CI* is similar to the *CI* based on *MLE*.

5. An Illustrative Example

In order to illustrate the results obtained in the preceding sections, we simulated two independent upper *RRSS*s with sizes $n = m = 7$, i.e. $\mathbf{r} = (r_{1,1}, ..., r_{7,7})^T$ and $\mathbf{s} = (s_{1,1}, ..., s_{7,7})^T$, from $Exp(\theta_1 = 7)$ and $Exp(\theta_2 = 3)$. The generated samples are displayed in Table 3. From Table 3, it is observed that $r_{i,i}$'s and $s_{i,i}$'s are not necessarily ordered, as mentioned earlier. The true value of the stress-strength R is equal to $\frac{\theta_2}{\theta_2+\theta_1} = 0.3$. The *MLE*s of the parameters θ_1 and θ_2 are obtained as 6.8994 and 2.7709, respectively. So, from (6) and (9), $\hat{R}_{ML}$ and $\hat{R}_{UMVU}$ are derived as 0.2865 and 0.2832, respectively. Therefore, both the *MLE* and the *UMVUE* of R are close to the true value. From (13) and (16) and also Algorithms 3.1-3.3, we derived the corresponding *CI*s for R. We got the number of bootstrap replication, B=1000, and presented the results in Table 4. From Table 4, it is observed that all *CI*s contain the true value of the stress-strength, i.e. R=0.3. Furthermore, as it is observed from the entries of Tables 1 and 2 (for $n = m = 7$ and R=0.3), the $Boot_2$-t is the longest *CI*. Thus the obtained results in this section confirm the results in the previous sections.

Table 3: The simulated data.

r	5.907	23.714	22.352	20.875	26.800	56.346	37.190
s	1.710	10.678	2.773	5.069	19.958	29.970	7.427

Table 4: The 100$(1-\alpha)$% *CI* for *R* based on the simulated data given by Table 3.

	$\alpha = 0.1$			$\alpha = 0.05$		
CI	Lower	Upper	Length	Lower	Upper	Length
Perc	0.2067	0.3857	0.1789	0.1991	0.4061	0.2070
$Boot_1$-t	0.2011	0.3862	0.1851	0.1852	0.3987	0.2135
$Boot_2$-t	0.1996	0.3900	0.1904	0.1750	0.4097	0.2347
MLE	0.2050	0.3849	0.1799	0.1913	0.4054	0.2141
AMLE	0.1967	0.3764	0.1797	0.1794	0.3936	0.2142

6. Conclusion

In this paper, we have obtained *MLE* as well as *UMVUE* for stress-strength parameter R on the basis of upper *RRSS* from the exponential distribution. These point estimators have been compared with respect to the *MSE* criterion. It is observed that *MLE* has better performance when R is close to 0.5 while *UMVUE* is better for the extreme values of R. Also, we derived an exact as well as an approximated *CI* based on *MLE* and then compared them with three bootstrap *CI*s. Based on a simulation result and an illustrative example we observed that the percentile *CI* and the exact *CI* of *MLE* have better performance than the other *CI*s.

Acknowledgment

The author would like to thank the anonymous referee for careful reading and many helpful suggestions.

References

[1] J. Ahmadi, N.R. Arghami, Some univariate stochastic orders on record values. *Communication in Statistics Theory Methods*, (2001) **30**, 69–74.
[2] J. Ahmadi, N.R. Arghami, Comparing the Fisher information in record values and iid observations. *Statistics* (2003) **37**, 435–441.
[3] Arnold, B. C., Balakrishnan, N. and Nagaraja, H. N. (1998). *Records*, John Wiley & Sons, New York.
[4] A. Baklizi, Estimation of $Pr(X < Y)$ using record values in the one and two parameter exponential distributions. *Communications in Statistics Theory and Methods*, (2008a) **37**, 692–698.
[5] A. Baklizi, Likelihood and Bayesian estimation of $Pr(X < Y)$ using lower record values from the generalized exponential distribution. *Computational Statistics and Data Analysis*, (2008b) **52**, 3468–3473.
[6] A. Baklizi, Interval estimation of the stress-strength reliability in the two-parameter exponential distribution based on records. *Journal of Statistical Computation and Simulation*, (2014) **84**, 2670–2679.
[7] N. Balakrishnan, A. P. Basu, (Eds.) *The Exponential Distribution; Theory, Methods and Applications*. Gordon & Breach Science Publishers, Newark, New Jersey (1995).
[8] N. Balakrishnan, U. Kamps, and Kateri, M. Minimal repair under a step-stress test. *Statistics & Probability Letters*, (2009) **79**, 1548–1558.
[9] Z. W. Birnbaum, B.C. McCarty, A distribution-free upper confidence bounds for $Pr(Y < X)$ based on independent samples of X and Y. *The Annals of Mathematical Statistics*, (1958) **29**, 558–562.
[10] M. Brown, F. Proschan, Imperfect repair. *Journal of Applied Probability*, (1983) **20**, 851–859.
[11] Z. Chen, , Z. Bai, B. K. Sinha, *Ranked Set Sampling: Theory and Applications*, Springer-Vergla, New York (2004).
[12] I. Dattner, Deconvolution of $P(X < Y)$ with supersmooth error distributions. *Statistics & Probability Letters*, (2013) **83**, 1880–1887.
[13] B. Efron, R. Tibshirani, *An Introduction to the Bootstrap*, Chapman and Hall, New York (1993).
[14] S. Erylmaz, On system reliability in stress-strength setup. *Statistics & Probability Letters*, (2010) **80**, 834–839.
[15] N. Glick, Breaking record and breaking boards. *The American Mathematical Monthly*, (1978) **85**, 2–26.
[16] S. Gulati, W. J. Padgett, *Parametric and Non-parametric Inference from Record-breaking Data*. Lecture Notes in Statistics, Vol. 172. (2003) Springer, New York.
[17] S. Kotz,, Y. Lumelskii, M. Pensky, *The Stress-Strength Model and Its Generalizations: Theory and Applications*, World Scientific, Singapore (2003).
[18] D. Kundu, M. Z. Raqab, Estimation of $R = P(Y < X)$ for three-parameter Weibull distribution. *Statistics & Probability Letters*, (2009) **79**, 1839–1846.
[19] Lehmann, E. L. and Cassela, G. (1998). *Theory of Point Estimation*. John Wiley, New York.
[20] Muttlak, H. A., Abu-Dayyeh, W. A., Saleh, M. F. and Al-Sawi, E. (2010). Estimating Pr($Y < X$) using ranked set sampling in case of the exponential distribution. *Communication in Statistics: Theory and Methods*, **39**, 1855–1868.
[21] Owen, D. B., Craswell, K. J. and Hanson, D. L. (1964). Non-parametric upper confidence bounds for $P(Y < X)$ and confidence limits for $P(Y < X)$ when X and Y are normal. *Journal of American Statistical Association*, **59**, 906–924.
[22] Rizzo, M. L. (2008). *Statistical Computing with R*, Chapman & Hall, Taylor & Francis Group, LLc.
[23] Salehi, M. and Ahmadi, J. (2014). Record ranked set sampling scheme. *Metron*, (2014) **72**, 351–365.
[24] Wasserman, L. (2006). *All of Non-parametric Statistics*, Springer, New York.